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On almost pseudo cyclic Ricci symmetric manifolds

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Abstract. The object of the present paper is to introduce a type of non-flat Riemannian manifolds called *almost pseudo cyclic Ricci symmetric manifold* and study its geometric properties. Among others it is shown that an almost pseudo cyclic Ricci symmetric manifold is a special type of quasi-Einstein manifold. We also study conformally flat almost pseudo cyclic Ricci symmetric manifolds and prove that such a manifold is isometrically immersed in an Euclidean manifold as a hypersurface. The existence of such notion is ensured by a non-trivial example.

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§1. Introduction

By extending the definition of pseudo Ricci symmetric manifold, very recently M. C. Chaki and T. Kawaguchi [8] introduced the notion of almost pseudo Ricci symmetric manifold. A Riemannian manifold $(M^n, g)(n > 2)$ is called an almost pseudo Ricci symmetric manifold if its Ricci tensor S of type $(0, 2)$ is not identically zero and satisfies the condition

$$(1.1)\qquad \begin{aligned}(\nabla_X S)(Y, Z) &= [A(X) + B(X)]S(Y, Z)\\ &+A(Y)S(X, Z) + A(Z)S(Y, X) \quad \text{for all } X,\ Y,\ Z \in \chi(M),\end{aligned}$$

where ∇ denotes the operator of covariant differentiation with respect to the metric tensor g and $\chi(M)$ is the Lie algebra of all smooth vector fields on M; and A and B are nowhere vanishing 1-forms associated with the unique vector field U and V respectively such that $g(X, U) = A(X)$ and $g(X, V) = B(X)$ for all $X \in \chi(M)$. The vector fields U, V are called the generators of the manifold.

The 1-forms A and B are called associated 1-forms and an n-dimensional manifold of this kind is denoted by $A(PRS)_n$. If, in particular, $B = A$ then (1.1) reduces to the notion of pseudo Ricci symmetric manifold [6]. Extending the notion of almost pseudo Ricci symmetric manifold, in the present paper we introduce the notion of *almost pseudo cyclic Ricci symmetric manifold.* A Riemannian manifold $(M^n, g)(n > 2)$ (this condition is assumed throughout the paper) is said to be an almost pseudo cyclic Ricci symmetric manifold if its Ricci tensor S of type (0, 2) is not identically zero and satisfies the following:

$$\begin{aligned}(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y)\\ = [A(X)+B(X)]S(Y,Z) + A(Y)S(X,Z) + A(Z)S(Y,X),\end{aligned} \tag{1.2}$$

where A and B are nowhere vanishing 1-forms associated to the unique vector field U and V respectively such that $A(X) = g(X,U)$ and $B(X) = g(X,V)$ for all X and ∇ denotes the operator of covariant differentiation with respect to the metric tensor g. Such an n-dimensional manifold is denoted by $A(PCRS)_n$.

The present paper is organized as follows. Section 2 is concerned with some basic properties of $A(PCRS)_n$. Every $A(PRS)_n$ is $A(PCRS)_n$ but not conversely. However, it is proved that an $A(PCRS)_n$ with Codazzi type Ricci tensor is an $A(PRS)_n$. Again it is shown that an $A(PCRS)_n$ is a special type of quasi-Einstein manifold ([7], [10], [11]) and also it is proved that the scalar curvature of an $A(PCRS)_n$ is always non-zero at every point of the manifold.

In Section 3 we investigate conformally flat $A(PCRS)_n$ and prove that such a manifold is of quasi-constant curvature. It is shown that in a conformally flat $A(PCRS)_n$, the unique unit vector field λ defined by $\frac{B(X)}{\sqrt{B(V)}} = T(X) = g(X,\lambda)$, for all $X \in \chi(M)$, is a unit proper concircular vector field and hence such a manifold is a subprojective manifold in the sense of Kagan ([1], [20]). Again it is proved that a conformally flat $A(PCRS)_n$ can be expressed as the warped product $I \times_{e^p} \overset{*}{M}$, where $(\overset{*}{M}, \overset{*}{g})$ is an $(n-1)$ dimensional Einstein manifold [13].

As a generalization of subprojective manifold B. Y. Chen and K. Yano [9] introduced the notion of special conformally flat manifold. It is shown that a conformally flat $A(PCRS)_n$ with non-constant and negative scalar curvature is a special conformally flat manifold and also it is proved that such a simply connected manifold can be isometrically immersed in an Euclidean manifold E^{n+1} as a hypersurface.

A non-zero vector X on a semi-Riemannian manifold M is said to be timelike (resp., non-spacelike, null, spacelike) if it satisfies $g(X,X) < 0$ (resp. ≤ 0, $= 0$, > 0) [15]. Since λ is a unit vector field on the Riemannian manifold $M = A(PCRS)_n$ with metric tensor g, it can be easily shown [15](p.148) that $\widetilde{g} = g - 2T \otimes T$ is a Lorentz metric on M. Hence λ becomes timelike so that the resulting Lorentz manifold $(M, \widetilde{g})$ is time-orientable. The last section deals with a non-trivial example of $A(PCRS)_n$.

§2. Some basic properties of $A(PCRS)_n$

This section deals with various basic geometric properties of $A(PCRS)_n$. Let Q be the symmetric endomorphism of the tangent bundle of the manifold corresponding to the Ricci tensor S, i.e., $g(QX, Y) = S(X, Y)$ for all vector fields X, Y.

Let $\{e_i : i = 1, 2, \cdots, n\}$ be an orthonormal basis of the tangent space at each point of the manifold. Then setting $Y = Z = e_i$ in (1.2) and then taking summation over i, $1 \leq i \leq n$, we obtain

$$2dr(X) = r[A(X) + B(X)] + 2A(QX), \tag{2.1}$$

where r is the scalar curvature of the manifold. Again contracting (1.2) with respect to Z and X, and then replacing Y by X in the resulting equation, we get

$$2dr(X) = rA(X) + 2A(QX) + B(QX). \tag{2.2}$$

From (2.1) and (2.2) it follows that

$$B(QX) = rB(X), \tag{2.3}$$

i,e.,

$$S(X, V) = rg(X, V). \tag{2.4}$$

This leads to the following:

Proposition 1. *The scalar curvature r of an $A(PCRS)_n$ is an eigenvalue of the Ricci tensor S corresponding to the eigenvector V.*

We now suppose that the Ricci tensor of an $A(PCRS)_n (n > 2)$ is of Codazzi type ([12],[16]). Then we have

$$(\nabla_X S)(Y, Z) = (\nabla_Y S)(Z, X) = (\nabla_Z S)(X, Y) \tag{2.5}$$

for all X, Y, Z and hence the defining condition (1.2) of an $A(PCRS)_n$ reduces to

$$\begin{aligned} &(\nabla_X S)(Y, Z) \\ &= [\bar{A}(X) + \bar{B}(X)]S(Y, Z) + \bar{A}(Y)S(Z, X) + \bar{A}(Z)S(X, Y), \end{aligned} \tag{2.6}$$

where $\bar{A}(X) = g(X, \frac{1}{3}U)$ and $\bar{B}(X) = g(X, \frac{1}{3}V)$ are nowhere vanishing 1-forms, and consequently the manifold is $A(PRS)_n$. This leads to the following:

Proposition 2. *If the Ricci tensor of an $A(PCRS)_n$ is of Codazzi type, then it is an $A(PRS)_n$.*

Interchanging X and Y in (1.2) we obtain

$$\begin{aligned}&(\nabla_Y S)(X,Z)+(\nabla_X S)(Z,Y)+(\nabla_Z S)(Y,X)\\&=[A(Y)+B(Y)]S(X,Z)+A(X)S(Y,Z)+A(Z)S(Y,X).\end{aligned}$$

By virtue of (1.2) and above equation, it follows that

$$B(X)S(Y,Z)=B(Y)S(X,Z)\ \forall X,\ Y,\ Z\in\chi(M). \tag{2.7}$$

Setting $Y=V$ in (2.7) we get

$$S(X,Z)=\frac{1}{B(V)}B(X)B(QZ). \tag{2.8}$$

In view of (2.3), (2.8) yields

$$S(X,Z)=rT(X)T(Z), \tag{2.9}$$

where $T(X)=g(X,\lambda)=\frac{1}{\sqrt{B(V)}}B(X)$, λ being a unit vector field associated with the nowhere vanishing 1-form T. From (2.9) it follows that if $r=0$ at every point of the manifold, then $S(X,Z)=0$, which is inadmissible by the definition of $A(PCRS)_n$. Hence we can state the following:

Proposition 3. *The scalar curvature of an $A(PCRS)_n$ is non-zero at every point of the manifold, and its Ricci tensor is of the form (2.9).*

A Riemannian manifold $(M^n,g)(n>2)$ is said to be quasi-Einstein manifold ([7], [10], [11]) if its Ricci tensor S of type (0, 2) is not identically zero and satisfies the following:

$$S(X,Y)=\alpha g(X,Y)+\beta D(X)D(Y), \tag{2.10}$$

where α,β are scalars of which $\beta\neq 0$ and D is a nowhere vanishing 1-form associated with the unique unit vector field σ defined by $g(X,\sigma)=D(X)$ for all $X\in\chi(M)$. Hence from (2.9) we can state the following:

Proposition 4. *An $A(PCRS)_n$ is a special type of quasi-Einstein manifold.*

If an $A(PCRS)_n(n>2)$ is Einstein, then from (1.2) it follows that

$$\{A(X)+B(X)\}g(Y,Z)+A(Y)g(Z,X)+A(Z)g(X,Y)=0, \tag{2.11}$$

as r is non-zero at every point of the manifold. From (2.11), we can easily get $B(X)=0$, which is inadmissible by the definition of $A(PCRS)_n$. Hence we can state the following:

Proposition 5. *There does not exist any Einstein $A(PCRS)_n(n>2)$.*

§3. Conformally flat $A(PCRS)_n$

A Riemannian manifold $(M^n, g)(n > 2)$ is said to be the manifold of quasi-constant curvature ([2], [3], [4], [5], [17]) if it is conformally flat and its curvature tensor R of type (0, 4) has the following form:

$$
\begin{aligned}
(3.1) \qquad R(X,Y,Z,W) &= a_1[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \\
&\quad + a_2[g(Y,Z)A(X)A(W) - g(X,Z)A(Y)A(W) \\
&\quad + g(X,W)A(Y)A(Z) - g(Y,W)A(X)A(Z)],
\end{aligned}
$$

where A is a nowhere vanishing 1-form and a_1, a_2 are scalars of which $a_2 \neq 0$. Let a Riemannian manifold $(M^n, g)(n > 3)$ be conformally flat. Then the curvature tensor R of type (0, 4) is of the following form:

$$
\begin{aligned}
(3.2) \qquad R(X,Y,Z,W) &= \frac{1}{n-2}[\{S(Y,Z)g(X,W) - S(X,Z)g(Y,W) \\
&\quad + S(X,W)g(Y,Z) - S(Y,W)g(X,Z)\} \\
&\quad - \frac{r}{(n-1)}\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\}].
\end{aligned}
$$

Using (2.9) in (3.2) we obtain

$$
\begin{aligned}
(3.3) \qquad R(X,Y,Z,W) &= a[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \\
&\quad + b[g(X,W)T(Y)T(Z) - g(Y,W)T(X)T(Z) \\
&\quad + g(Y,Z)T(X)T(W) - g(X,Z)T(Y)T(W)],
\end{aligned}
$$

where $a = -\frac{r}{(n-1)(n-2)}$ and $b = \frac{r}{n-2}$ are non-zero scalars. By virtue of (3.1), it follows from (3.3) that a conformally flat $A(PCRS)_n(n > 3)$ is a manifold of quasi-constant curvature. This leads to the following:

Theorem 1. *Every conformally flat $A(PCRS)_n(n > 3)$ is a manifold of quasi-constant curvature.*

Since in a 3-dimensional Riemannian manifold the conformal curvature tensor C vanishes, the relation (3.2) holds and hence (3.3) also holds. This leads to the following:

Corollary 1. *Every $A(PCRS)_3$ is a manifold of quasi-constant curvature.*

The conformal curvature tensor C of type (1, 3) of a Riemannian manifold (M^n, g) $(n > 3)$ is defined by

$$
\begin{aligned}
(3.4) \qquad C(X,Y)Z &= R(X,Y)Z - \frac{1}{n-2}[S(Y,Z)X \\
&\quad - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] \\
&\quad + \frac{r}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y].
\end{aligned}
$$

Differentiating (3.4) covariantly and then contracting we obtain

$$
\begin{aligned}
(3.5)(divC)(X,Y)Z &= (divR)(X,Y)Z - \frac{1}{n-2}[(\nabla_X S)(Y,Z) \\
&\quad -(\nabla_Y S)(X,Z) + \frac{1}{2}dr(X)g(Y,Z) - \frac{1}{2}dr(Y)g(X,Z)] \\
&\quad + \frac{1}{(n-1)(n-2)}[dr(X)g(Y,Z) - dr(Y)g(X,Z)],
\end{aligned}
$$

where 'div' denotes the divergence. Again it is known that in a Riemannian manifold, we have

$$(divR)(X,Y)Z = (\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z).$$

Consequently by virtue of the above relation, (3.5) takes the form

$$
\begin{aligned}
(3.6)\quad (divC)(X,Y)Z &= \frac{n-3}{n-2}[(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) \\
&\quad - \frac{1}{2(n-1)}\{dr(X)g(Y,Z) - dr(Y)g(X,Z)\}].
\end{aligned}
$$

Let us consider a conformally flat $A(PCRS)_n (n > 3)$. Then we have

$$(divC)(X,Y)Z = 0$$

and hence (3.6) yields

$$
\begin{aligned}
(3.7)\qquad & (\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) \\
&= \frac{1}{2(n-1)}[dr(X)g(Y,Z) - dr(Y)g(X,Z)].
\end{aligned}
$$

Again from (2.9) we have

$$
\begin{aligned}
(3.8)\qquad & (\nabla_Y S)(X,Z) \\
&= dr(Y)T(X)T(Z) + r[(\nabla_Y T)(X)T(Z) + T(X)(\nabla_Y T)(Z)].
\end{aligned}
$$

Using (3.8) in (3.7) we obtain

$$
\begin{aligned}
(3.9)\qquad & dr(X)T(Y)T(Z) - dr(Y)T(X)T(Z) + r[(\nabla_X T)(Y)T(Z) \\
& +T(Y)(\nabla_X T)(Z) - (\nabla_Y T)(X)T(Z) - T(X)(\nabla_Y T)(Z)] \\
&= \frac{1}{2(n-1)}[dr(X)g(Y,Z) - dr(Y)g(X,Z)].
\end{aligned}
$$

Setting $Y = Z = e_i$ in (3.9) and taking summation over $i, 1 \le i \le n$, we get

$$(3.10)\qquad dr(\lambda)T(X) + r\{(\nabla_\lambda T)(X) + T(X)\sum_{i=1}^{n}(\nabla_{e_i} T)(e_i)\} = \frac{1}{2}dr(X).$$

Again putting $Y = Z = \lambda$ in (3.9) we obtain

$$r(\nabla_\lambda T)(X) = \frac{2n-3}{2(n-1)}[dr(X) - dr(\lambda)T(X)]. \tag{3.11}$$

Using (3.11) in (3.10) we get

$$rT(X)\sum_{i=1}^{n}(\nabla_{e_i}T)(e_i) + \frac{n-2}{2(n-1)}dr(X) + \frac{1}{2(n-1)}dr(\lambda)T(X) = 0. \tag{3.12}$$

Setting $X = \lambda$ in (3.12) we get

$$r\sum_{i=1}^{n}(\nabla_{e_i}T)(e_i) = -\frac{1}{2}dr(\lambda). \tag{3.13}$$

From (3.12) and (3.13), it follows that

$$dr(X) = dr(\lambda)T(X). \tag{3.14}$$

Again plugging $Z = \lambda$ in (3.9) and then using (3.14), we obtain

$$r\{(\nabla_X T)(Y) - (\nabla_Y T)(X)\} = 0,$$

which implies that

$$(\nabla_X T)(Y) - (\nabla_Y T)(X) = 0, \tag{3.15}$$

since $r \neq 0$. The relation (3.15) implies that the 1-form T is closed. In view of (3.14) it follows from (3.11) that

$$(\nabla_\lambda T)(Z) = 0, \tag{3.16}$$

which implies that $\nabla_\lambda \lambda = 0$. The existence of the integral curve of λ is ensured by Proposition 49 of ([15], p. 28). Hence we can state the following:

Theorem 2. *In a conformally flat $A(PCRS)_n (n > 3)$, the integral curves of the vector field λ are geodesics.*

Also setting $Y = \lambda$ in (3.9) we obtain by virtue of (3.14) and (3.16) that

$$(\nabla_X T)(Z) = \frac{1}{2(n-1)r}dr(\lambda)[T(X)T(Z) - g(X,Z)]. \tag{3.17}$$

Let us now consider a non-zero scalar function $f = \frac{1}{2(n-1)r}dr(\lambda)$, where the scalar curvature r is non-constant. Then we have

$$\nabla_X f = -\frac{1}{2(n-1)r^2}dr(\lambda)dr(X) + \frac{1}{2(n-1)r}d^2r(\lambda, X). \tag{3.18}$$

From (3.14) it follows that

$$d^2r(X,Y) = d^2r(\lambda,Y)T(X) + dr(\lambda)(\nabla_Y T)(X). \tag{3.19}$$

Since a second covariant differential of any function is a symmetric 2-form, (3.15) and (3.19) give

$$d^2r(\lambda,Y)T(X) = d^2r(\lambda,X)T(Y). \tag{3.20}$$

Replacing Y by λ in (3.20) we have

$$d^2r(\lambda,X) = d^2r(\lambda,\lambda)T(X) = -\psi T(X), \tag{3.21}$$

where $\psi = -d^2r(\lambda,\lambda)$ is a scalar function. Using (3.14) and (3.21) in (3.18) we obtain

$$\nabla_X f = \mu T(X), \tag{3.22}$$

where

$$\mu = -\frac{1}{2(n-1)r^2}[r\psi + \{dr(\lambda)\}^2]$$

is a scalar. Since r is non-zero and non-constant function, $dr(X) \neq 0$ for all X and hence $dr(\lambda) \neq 0$, and therefore $\psi \neq 0$. Consequently μ is a non-zero scalar. We now consider an 1-form ω defined by

$$\omega(X) = \frac{1}{2(n-1)r}dr(\lambda)T(X) = fT(X).$$

Then by virtue of (3.15) and (3.22) we have

$$d\omega(X,Y) = 0.$$

Hence the 1-form ω is closed. Therefore (3.17) can be rewritten as

$$(\nabla_X T)(Z) = -fg(X,Z) + \omega(X)T(Z), \tag{3.23}$$

which implies that the vector field λ is a proper unit concircular vector field ([18], [19]). Again in a conformally flat 3-dimensional Riemannian manifold, the relation (3.7) holds for $n = 3$. Hence proceeding similarly as above it is easy to check that in an $A(PCRS)_3$, λ is a unit proper concircular vector field. Thus we can state the following:

Theorem 3. *In a conformally flat $A(PCRS)_n$ with non-constant scalar curvature the vector field λ defined by $g(X,\lambda) = T(X)$ is a unit proper concircular vector field.*

Again if a conformally flat Riemannian manifold $(M^n, g), n > 3$, admits a proper concircular vector field, then the manifold is a subprojective manifold of Kagan ([1], [20]) and hence by virtue of Theorem 3, we can state the following:

Theorem 4. *Every conformally flat $A(PCRS)_n, n > 3$, with non-constant scalar curvature is a subprojective manifold in the sense of Kagan.*

In [19] K. Yano proved that a Riemannian manifold admits a concircular vector field, if and only if there exists a coordinate system with respect to which the fundamental quadratic differential form can be written as

$$ds^2 = dx'^2 + e^p \overset{*}{g}_{ij} \, dx^i dx^j,$$

where $\overset{*}{g}_{ij} = g_{ij}[x^k]$ are the functions of x^k only $(i, j, k = 2, 3, \cdots, n)$ and $p = p(x')$ is a non-constant function of x' only. Hence if an $A(PCRS)_n, n > 3$, is conformally flat, then it is a warped product $I \times_{e^p} \overset{*}{M}$, where $(\overset{*}{M}, \overset{*}{g})$ is an $(n-1)$ dimensional Riemannian manifold, and $I \subset \mathbf{R}$ is an open interval. Again, A. Gebarowski [13] proved that the warped product $I \times_{e^p} \overset{*}{M}$ satisfies the condition (3.7) if and only if $\overset{*}{M}$ is an Einstein manifold. Since a conformally flat $A(PCRS)_n$ satisfies (3.7), it must be a warped product $I \times_{e^p} \overset{*}{M}$, where $\overset{*}{M}$ is an Einstein manifold. Hence we can state the following:

Theorem 5. *Every conformally flat $A(PCRS)_n, n > 3$, with non-constant scalar curvature can be expressed as a warped product $I \times_{e^p} \overset{*}{M}$, where $(\overset{*}{M}, \overset{*}{g})$ is an $(n-1)$ dimensional Einstein manifold.*

As a generalization of subprojective manifold, B. Y. Chen and K. Yano [9] introduced the notion of special conformally flat manifold defined as follows: a conformally flat Riemannian manifold is said to be a special conformally flat manifold if the tensor field H of type (0, 2) defined by

$$H(X, Y) = -\frac{1}{n-2} S(X, Y) + \frac{r}{2(n-1)(n-2)} g(X, Y) \tag{3.24}$$

is expressible in the form

$$H(X, Y) = -\frac{\alpha^2}{2} g(X, Y) + \beta(X\alpha)(Y\alpha), \tag{3.25}$$

where α, β are two scalars such that α is positive. In view of (2.9), (3.24) can be written as

$$H(X, Y) = -\frac{r}{n-2} T(X)T(Y) + \frac{r}{2(n-1)(n-2)} g(X, Y). \tag{3.26}$$

Let us now take

$$\alpha^2 = -\frac{r}{(n-1)(n-2)} > 0, \quad \text{provided } r < 0. \tag{3.27}$$

Then

$$2\alpha(X\alpha) = -\frac{dr(X)}{(n-1)(n-2)},$$

which implies by virtue of (3.14) that

$$2\alpha(X\alpha) = -\frac{dr(\lambda)T(X)}{(n-1)(n-2)}.$$

Hence

$$T(X)T(Y) = -\frac{4(n-1)(n-2)r}{\delta^2}(X\alpha)(Y\alpha),$$

where $\delta = dr(\lambda) \neq 0$. Therefore, by virtue of (3.27), (3.26) can be expressed as

$$H(X,Y) = -\frac{\alpha^2}{2}g(X,Y) + \beta(X\alpha)(Y\alpha),$$

where $\beta = \frac{4(n-1)r^2}{\delta^2} \neq 0$ at each point of the manifold, provided r is non-constant. Hence the manifold is a special conformally flat manifold. This leads to the following:

Theorem 6. *A conformally flat $A(PCRS)_n, n > 3$, with non-constant and negative scalar curvature is a special conformally flat manifold.*

Again in [9] it is proved that every simply-connected special conformally flat manifold can be isometrically immersed in an Euclidean manifold E^{n+1} as a hypersurface. Therefore by virtue of Theorem 6, we can state the following:

Theorem 7. *Every simply-connected conformally flat $A(PCRS)_n, n > 3$, with non-constant and negative scalar curvature can be isometrically immersed in an Euclidean manifold E^{n+1} as a hypersurface.*

§4. An example of $A(PCRS)_4$

Example . Let $M = \mathbf{R}^3 \times (0,\infty) \subseteq \mathbf{R}^4$. We take the identity map I_d on M such that $N = I_d(M) = \mathbf{R}^3 \times (0,\infty)$. Then $\{E_1 = (1,0,0,0)$, $E_2 = (0,1,0,0)$, $E_3 = (0,0,1,0)$, $E_4 = (0,0,0,1)\}$ form a basis at each point $p \in N$. We define the Lorentz metric g on N by giving its components relative to the basis $\{E_i(p)\}$ at each point $p = (x^1,x^2,x^3,x^4)$ as follows:

$$g_{ij}(p) = g_{ij}(x^1,x^2,x^3,x^4) = \begin{cases} 0 & \text{if } \ i \neq j; \\ (x^4)^{\frac{4}{3}} & \text{if } \ i = j = 1,2,3; \\ -1 & \text{if } \ i = j = 4, \end{cases} \tag{4.1}$$

where $x^i, i = 1, 2, 3, 4$ are the standard coordinates of $\mathbf{R}^4$. Then the manifold N with metric (4.1) represents the Einstein-deSitter spacetime which represents the simplest cosmological model in general relativity ([14], p. 89). Hence for $X, Y \in T_pN$,

$$\begin{aligned} g_p(X,Y) &= g_p(X^iE_i, Y^jE_j) \\ &= (x^4)^{\frac{4}{3}}(X^1Y^1 + X^2Y^2 + X^3Y^3) - X^4Y^4, \end{aligned}$$

where x^4 is the 'height' of p. We note that this is a Lorentz metric since smoothness is obvious on $x^4 > 0$ and, for each $p \in N$, one can define a new basis $\{e_i : i = 1, 2, 3, 4\}$ for T_pN by $e_i = (x^4)^{-\frac{2}{3}}E_i$ for $i = 1, 2, 3$ and $e_4 = E_4$. Then $g(e_i, e_j) = \eta_{ij}$, where

$$\eta_{ij} = \begin{cases} 1 & \text{for} \quad i = j = 1, 2, 3; \\ -1 & \text{for} \quad i = j = 4; \\ 0 & \text{otherwise.} \end{cases}$$

Also we observe that for $p \in N$ the null cone at p is

$$\left\{X = X^iE_i \in T_pN : (x^4)^{\frac{4}{3}}((X^1)^2 + (X^2)^2 + (X^3)^2) = (X^4)^2\right\}$$

which one might interpret geometrically as saying that the null cones in N 'get steeper' as p 'gets higher'. Then the only non-zero Christoffel symbols for the standard coordinate patch on N are given by ([14], p. 95, Exercise 3.3.4)

$$\Gamma^1_{14} = \Gamma^1_{41} = \Gamma^2_{24} = \Gamma^2_{42} = \Gamma^3_{34} = \Gamma^3_{43} = \frac{2}{3(x^4)} \tag{4.2}$$

and the components of the Ricci tensor relative to the standard coordinate patch are given by

$$S_{ij} = \begin{cases} \frac{2}{3(x^4)^2} & \text{for } i = j = 4, \\ 0 & \text{otherwise.} \end{cases}$$

Also the scalar curvature of N is given by

$$r = -\frac{2}{3(x^4)^2} \neq 0.$$

In terms of local coordinates the defining condition (1.2) of an $A(PCRS)_4$ can be written as

$$\begin{aligned} &S_{ij,k} + S_{jk,i} + S_{ki,j} \\ &= (A_k + B_k)S_{ij} + A_iS_{jk} + A_jS_{ik}, \quad i, j, k = 1, 2, 3, 4. \end{aligned} \tag{4.3}$$

In view of (4.2) and (4.3) it follows that

$$S_{ij,\,k} = \begin{cases} -\frac{4}{3(x^4)^3} & \text{for } i=j=k=4, \\ 0 & \text{otherwise.} \end{cases}$$

In terms of local coordinates, if we consider the components of the 1-forms A and B as follows:

$$A_i = \begin{cases} -\frac{4}{3x^4} & \text{for } \quad i=4, \\ 0 & \text{otherwise ;} \end{cases} \tag{4.4}$$

$$B_i = \begin{cases} -\frac{2}{x^4} & \text{for } \quad i=4, \\ 0 & \text{otherwise ;} \end{cases} \tag{4.5}$$

then in view of (4.4) - (4.5), it is easy to check that (4.3) holds for all $i, j, k = 1, 2, 3, 4$. Therefore N equipped with metric (4.1) is an $A(PCRS)_4$ which is not $A(PRS)_4$. This leads to the following:

Theorem 8. *An Einstein-deSitter spacetime N equipped with the metric (4.1) is an $A(PCRS)_4$ which is not an $A(PRS)_4$.*

Remark 1. *The metric defined in (4.1) is not positive definite.*

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